

**ON THE RELATION OF FOUR-DIMENSIONAL $N=2,4$ –
SUPERSYMMETRIC STRING BACKGROUNDS TO INTEGRABLE
MODELS**

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ABSTRACT

In this letter we discuss the relation of four-dimensional, $N = 2$ supersymmetric string backgrounds to integrable models. In particular we show that non-Kählerian gravitational backgrounds with one $U(1)$ isometry plus non-trivial antisymmetric tensor and dilaton fields arise as the solutions of the Liouville equation or, for the case of vanishing central charge deficit, as the solutions of the continual Toda equation. When performing an Abelian duality transformation, a particular class of solutions of the continual Toda equation leads to the well-known gravitational Eguchi-Hanson instanton background with self-dual curvature tensor.

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Recently, the investigation of four-dimensional non-trivial backgrounds for consistent string propagation has attracted a lot of attention [1, 2, 3]. This research is essentially the continuation of the long-existing program of finding solutions to vacuum Einstein gravity in general relativity in the sense that in string theory a specific form of matter, namely the dilaton (Φ) plus antisymmetric tensor field ($B_{\mu\nu}$) system, is added. Of particular interest, in general relativity as well as in string theory, are those background configurations which exhibit some Killing symmetries. The reasoning for this is twofold. First, the existence of symmetries often implies that the background field equations are exactly solvable since the symmetries effectively reduce the theory to a lower dimensional system. In this way the field equations are very often related to integrable systems as demonstrated for the case of pure gravity in many examples. Second, the isometries can be used to generate new solutions starting from a given solved model. Particular famous examples in general relativity are the Ehlers and Geroch transformations [4] which relate different gravitational backgrounds to each other. In string theory, these transformations are extended in the sense that they generically mix the gravitational background with the Φ , $B_{\mu\nu}$ matter system. Specifically, for the case of an Abelian $U(1)^d$ isometry, the string solution generating transformations are elements of the group $O(d, d, R)$ [5, 6, 7]. In fact, one can even formulate a string Geroch group as a current algebra extension [8] containing as its ‘zero mode’ part not only the $O(d, d)$ transformations but also the S -field duality transformations [9]. Moreover one can show that in string theory the discrete subset of $O(d, d, Z)$ transformations, the so-called target space duality transformations [10], leave the underlying conformal field theory invariant. Therefore the duality transformations generically relate geometrically or even topologically [11] different background spaces which are nevertheless equivalent from the string point of view.

Of particular importance are those backgrounds which are consistent with (extended) world sheet supersymmetries and allow for the construction of an (extended) superconformal algebra. Thus they provide consistent backgrounds for heterotic or type II superstrings and are expected to lead also to (extended) supersymmetries in the target space-time. In [2] a systematic discussion on four-dimensional backgrounds with $N = 2$ world sheet supersymmetry was given. There, a set of conditions, imposed by world-sheet $N = 2$ supersymmetry, was derived for finding Kählerian as well as non-Kählerian four-dimensional (non-compact) target spaces. For example, a broad class of non-Kählerian target spaces with torsion could be constructed as solutions to a very simple integrable system, namely the one having the Laplace equation as field equation. In addition, these backgrounds have vanishing central charge deficit δc , and one therefore expects to be dealing with an enhanced $N = 4$ superconformal symmetry. Specifically the so-called

axionic instantons [12], which satisfy a self-duality equation in the dilaton plus axion matter system, fall into this class. The dual of the axionic instantons are given by a pure gravitational, Kählerian background having the form of a cosmological string [13]. Also the gravitational wave plus $B_{\mu\nu}$ background of ref.[3] can be shown [14] to follow from the Laplace equation, and the corresponding currents of the $N = 4$ superconformal algebra can be explicitly constructed [14].

In the following, a new $N = 2$ supersymmetric four-dimensional solution of the non-Kählerian type, i.e. with non-trivial $B_{\mu\nu}$ and Φ , is presented. Due to its $U(1)$ isometry this new solution is again related to an integrable model, namely it is shown to satisfy an equation of the Liouville type and, in the case of a vanishing central charge deficit, $\delta c = 0$, also the continual Toda equation recently discussed in [15] in the context of self-dual, purely gravitational backgrounds. For a vanishing central charge deficit, $\delta c = 0$, this new solution describes a four-dimensional target space which is dual to a Ricci flat gravitational background describing an Eguchi-Hanson instanton. Since it is known that the Eguchi-Hanson background is consistent with $N = 4$ world sheet supersymmetry, this then provides strong evidence that also the original non-Kählerian background with $\delta c = 0$ is $N = 4$ supersymmetric (assuming that the $N = 4$ world sheet supersymmetry doesn't get destroyed by T -duality transformations).

We begin with a brief review of some of the relevant aspects presented in [2] for the construction of non-trivial four-dimensional backgrounds with torsion. The most general $N = 2$ superspace action for one chiral superfield U and one twisted chiral superfield V in two dimensions is determined by a single real function $K(U, \bar{U}, V, \bar{V})$ [16]

$$S = \frac{1}{2\pi\alpha'} \int d^2x D_+ D_- \bar{D}_+ \bar{D}_- K(U, \bar{U}, V, \bar{V}) \quad (1)$$

The fields U and V obey chiral and twisted chiral constraints given, respectively, by

$$\bar{D}_\pm U = 0, \quad \bar{D}_+ V = D_- V = 0 \quad (2)$$

The target space interpretation of the theory can be made manifest by writing down the purely bosonic part of the superspace action (1)

$$S = -\frac{1}{2\pi\alpha'} \int d^2x [K_{u\bar{u}} \partial^a u \partial_a \bar{u} - K_{v\bar{v}} \partial^a v \partial_a \bar{v} + \epsilon_{ab} (K_{u\bar{v}} \partial_a u \partial_b \bar{v} + K_{v\bar{u}} \partial_a v \partial_b \bar{u})] \quad (3)$$

where

$$K_{u\bar{u}} = \frac{\partial^2 K}{\partial U \partial \bar{U}}, \quad K_{v\bar{v}} = \frac{\partial^2 K}{\partial V \partial \bar{V}}, \quad K_{u\bar{v}} = \frac{\partial^2 K}{\partial U \partial \bar{V}}, \quad K_{v\bar{u}} = \frac{\partial^2 K}{\partial V \partial \bar{U}} \quad (4)$$

Here, u and v denote the lowest components of the superfields U and V . Thus, the first two terms in (3) describe, in complex coordinates, the metric background of the model

given by

$$G_{\mu\nu} = \begin{pmatrix} 0 & K_{u\bar{u}} & 0 & 0 \\ K_{u\bar{u}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -K_{v\bar{v}} \\ 0 & 0 & -K_{v\bar{v}} & 0 \end{pmatrix}. \quad (5)$$

To obtain a space with Euclidean signature, one has to require $K_{u\bar{u}}$ to be positive definite whereas $K_{v\bar{v}}$ has to be negative definite. Note that the metric (5) is not Kähler.

The ϵ_{ab} -term in (3) provides the antisymmetric tensor field background

$$B_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & K_{u\bar{v}} \\ 0 & 0 & K_{v\bar{u}} & 0 \\ 0 & -K_{v\bar{u}} & 0 & 0 \\ -K_{u\bar{v}} & 0 & 0 & 0 \end{pmatrix}. \quad (6)$$

It follows that the field strength $H_{\mu\nu\lambda}$

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}, \quad (7)$$

can also be expressed entirely in terms of the function K as

$$H_{u\bar{u}v} = \frac{\partial^3 K}{\partial U \partial \bar{U} \partial V}, \quad H_{u\bar{u}\bar{v}} = -\frac{\partial^3 K}{\partial U \partial \bar{U} \partial \bar{V}}, \quad H_{v\bar{v}u} = \frac{\partial^3 K}{\partial V \partial \bar{V} \partial U}, \quad H_{v\bar{v}\bar{u}} = -\frac{\partial^3 K}{\partial V \partial \bar{V} \partial \bar{U}} \quad (8)$$

In order to have that the non-Kählerian background given by (5) and (6) provides a consistent string solution, the string equations of motion have to be satisfied. These are obtained by requiring the vanishing of the β -function equations, which at the one-loop level are given by [17]

$$\begin{aligned} 0 &= \beta_{\mu\nu}^G = R_{\mu\nu} - \frac{1}{4} H_\mu^{\lambda\sigma} H_{\nu\lambda\sigma} + 2\nabla_\mu \nabla_\nu \Phi + O(\alpha') \\ 0 &= \beta_{\mu\nu}^B = \nabla_\lambda H_{\mu\nu}^\lambda - 2(\nabla_\lambda \Phi) H_{\mu\nu}^\lambda + O(\alpha') \end{aligned} \quad (9)$$

$\Phi(u, \bar{u}, v, \bar{v})$ denotes the dilaton field. The central charge deficit δc for the gravitational background is determined by the vanishing of the β -function of the dilaton field as

$$\delta c \equiv c - \frac{3D}{2} = \frac{3}{2} \alpha' [4(\nabla\Phi)^2 - 4\nabla^2\Phi - R + \frac{1}{12}H^2] + O(\alpha'^2) \quad (10)$$

δc must be zero in order that the $N = 2$ world sheet supersymmetry is extended to $N = 4$. Then, in case of $N = 4$, the one-loop β -functions do not get higher order corrections and the lowest order solution is an exact superconformal field theory.

The equations of motion (9) and (10) for the non-Kählerian background (5) and (6) were explicitly worked out in [2]. There, it was found that the following quantities, $\mathcal{U} = \ln K_{u\bar{u}}$ and $\mathcal{V} = \ln K_{v\bar{v}}$, have to satisfy the following set of differential equations

$$\begin{aligned}\partial_u \mathcal{V} &= 2\partial_u \Phi + \bar{C}_1(\bar{u})e^{\mathcal{U}} \\ \partial_v \mathcal{U} &= 2\partial_v \Phi + \bar{C}_2(\bar{v})e^{\mathcal{V}}\end{aligned}\tag{11}$$

Here, $\bar{C}_1(u)$ and $\bar{C}_2(v)$ denote two apriori arbitrary holomorphic functions which arise as integration "constants" when solving the β -function equations (9). In order to proceed with the solving of (9) and (10), the following exclusive cases were then considered: (i) $C_1 = C_2 = 0$; (ii) $C_1 = 0, C_2 \neq 0$; (iii) $C_1, C_2 \neq 0$. We will, in the following, only be concerned with case (ii). It is for this case that we will construct a new non-trivial solution, which for a vanishing central deficit (10) will turn out to be the dual of the Eguchi-Hanson instanton.

As shown in [2], when considering case (ii) it is useful to perform the following holomorphic change of coordinates

$$w = \int \frac{dv}{C_2(v)}\tag{12}$$

Then, it can be shown that (9) implies that $\mathcal{V} = \mathcal{V}(u, \bar{u}, w + \bar{w})$, $\mathcal{U} = \mathcal{U}(u, \bar{u}, w + \bar{w})$ as well as $\Phi = \Phi(u, \bar{u}, w + \bar{w})$. Thus, case (ii) necessarily leads to at least one $U(1)$ Killing symmetry. The string equations of motion (9) can then for case (ii) finally be shown to be equivalent to the following differential equation for the Kähler potential $K(u, \bar{u}, w + \bar{w})$

$$\partial_w e^{K_w} = e^{c_1(w+\bar{w})+c_2} K_{u\bar{u}}\tag{13}$$

as well as to

$$2\Phi = \ln K_{w\bar{w}} - c_1(w + \bar{w}) + \text{constant}\tag{14}$$

Equation (14) defines the dilaton field $\Phi = \Phi(u, \bar{u}, w + \bar{w})$. c_1 and c_2 denote two arbitrary constants. Finally, the central charge deficit (10) is proportional to the constant c_1 appearing in (13), namely

$$\delta c = -2c_1\tag{15}$$

We now proceed to construct a non-trivial solution of (13) for the Kähler potential K . Once this is achieved, it is then straightforward to obtain the associated four-dimensional gravitational background through (5), (6) and (14). We begin by making an ansatz for $K_{u\bar{u}}$ as follows

$$K_{u\bar{u}} = \Upsilon(u, \bar{u}) \Omega(w + \bar{w})\tag{16}$$

As we will see, this separation ansatz reduces the field equations to a two-dimensional differential equation for Υ . Then, integrating (16) with respect to u and \bar{u} yields

$$K = \Omega(w + \bar{w}) \int dud\bar{u} \Upsilon(u, \bar{u}) + h(w + \bar{w}) (z(u) + \bar{z}(\bar{u})) + r(w + \bar{w}) \quad (17)$$

where the functions $h(w + \bar{w})$, $z(u)$ and $r(w + \bar{w})$ denote integration "constants". Inserting (16) into (13) and integrating once with respect to $w + \bar{w}$, on the other hand, yields

$$K = a(u, \bar{u}) + \int^{w+\bar{w}} \ln \left(f(u, \bar{u}) + \Upsilon(u, \bar{u}) \int^{w+\bar{w}} e^{c_1(w+\bar{w})+c_2} \Omega(w + \bar{w}) \right) \quad (18)$$

where the functions $a(u, \bar{u})$ and $f(u, \bar{u})$ denote two additional integration "constants". Equating (17) and (18) shows that compatibility of both can be achieved by demanding that

$$f(u, \bar{u}) = \alpha \Upsilon(u, \bar{u}) \quad (19)$$

where α denotes an arbitrary constant. Then, (18) turns into

$$K = a(u, \bar{u}) + (w + \bar{w}) \ln \Upsilon(u, \bar{u}) + \int^{w+\bar{w}} \ln \left(\alpha + \int^{w+\bar{w}} e^{c_1(w+\bar{w})+c_2} \Omega(w + \bar{w}) \right) \quad (20)$$

Hence, it follows by inspection that

$$r(w + \bar{w}) = \int^{w+\bar{w}} \ln \left(\alpha + \int^{w+\bar{w}} e^{c_1(w+\bar{w})+c_2} \Omega(w + \bar{w}) \right) \quad (21)$$

$$a(u, \bar{u}) + (w + \bar{w}) \ln \Upsilon(u, \bar{u}) = \Omega(w + \bar{w}) \int dud\bar{u} \Upsilon(u, \bar{u}) + h(w + \bar{w}) (z(u) + \bar{z}(\bar{u}))$$

There are now two possibilities. Either $\Omega = \text{const}$ or $\Omega \neq \text{const}$. We will in the following take $\Omega \neq \text{const}$. Then, it follows from (21) that

$$a(u, \bar{u}) = 0$$

$$(w + \bar{w}) \ln \Upsilon(u, \bar{u}) = \Omega(w + \bar{w}) \int dud\bar{u} \Upsilon(u, \bar{u}) + h(w + \bar{w}) (z(u) + \bar{z}(\bar{u})) \quad (22)$$

Now, differentiating (22) with respect to u and \bar{u} yields

$$(w + \bar{w}) \partial_u \partial_{\bar{u}} \ln \Upsilon(u, \bar{u}) = \Omega(w + \bar{w}) \Upsilon(u, \bar{u}) \quad (23)$$

Thus, it follows from (22) and (23) that

$$\Omega(w + \bar{w}) = \gamma (w + \bar{w})$$

$$h(w + \bar{w}) = \tau (w + \bar{w}) \quad (24)$$

where γ and τ are two arbitrary constants. It then also follows from (23) that $\ln \Upsilon$ has to satisfy the Liouville equation

$$\partial_u \partial_{\bar{u}} \ln \Upsilon(u, \bar{u}) = \gamma \Upsilon(u, \bar{u}) \quad (25)$$

The general solution to the Liouville equation (25) is well known and reads

$$\Upsilon(u, \bar{u}) = \frac{\partial_u F(u) \partial_{\bar{u}} G(\bar{u})}{(1 - \frac{\gamma}{2} FG)^2} \quad (26)$$

where $F(u)$ and $G(\bar{u})$ are arbitrary functions of u and \bar{u} , respectively. Acting with $SL(2, C)$ transformations on the functions $F(u)$ and $G(\bar{u})$ leaves Υ invariant and therefore does not change the background.

Thus, we have constructed a non-trivial solution to (13) for the Kähler potential K . K can now be directly read off from (20) and reads

$$K = (w + \bar{w}) \ln \Upsilon(u, \bar{u}) + \int^{w+\bar{w}} \ln \left(\alpha + \gamma \int^{w+\bar{w}} e^{c_1(w+\bar{w})+c_2} (w + \bar{w}) \right) \quad (27)$$

where, again, Υ is the solution (26) to the Liouville equation. The dilaton field takes the following form

$$2\Phi = \log \left(\frac{\gamma(w + \bar{w})}{\alpha + \gamma \int^{w+\bar{w}} e^{c_1(w+\bar{w})+c_2} (w + \bar{w})} \right) + constant \quad (28)$$

In the following, we will now specialise to the case where the central charge deficit (15) vanishes, that is when $c_1 = 0$. Then the theory is expected to possess $N = 4$ world sheet supersymmetry. We will also set $c_2 = i\pi$. Then, the Kähler potential (27) is evaluated to be

$$K = (w + \bar{w}) \ln \Upsilon(u, \bar{u}) + \int^{w+\bar{w}} \ln \left(\alpha - \frac{\gamma}{2} (w + \bar{w})^2 \right) \quad (29)$$

Due to the Liouville nature of Υ it can now be easily shown that the differential equation (13) for the Kähler potential (29) is related to the continual Toda equation recently discussed in [15], as follows.² Introducing $\Sigma = K_w$, it follows from (29) and (25) that

$$\partial_u \partial_{\bar{u}} \Sigma = \gamma \Upsilon(u, \bar{u}) \quad (30)$$

Differentiating (13) once with respect to $w + \bar{w}$ then indeed yields the continual Toda equation

$$\partial_w^2 e^\Sigma = -\partial_u \partial_{\bar{u}} \Sigma \quad (31)$$

²Actually, ref.[15] discussed the continual Toda equation for Kählerian backgrounds with self-dual metric.

The line element associated with (29) is readily computed to be

$$\begin{aligned} ds^2 &= -K_{w\bar{w}} dw d\bar{w} + K_{u\bar{u}} du d\bar{u} \\ &= \frac{\gamma(w + \bar{w})}{\alpha - \frac{\gamma}{2}(w + \bar{w})^2} dw d\bar{w} + \gamma(w + \bar{w}) \frac{\partial_u F(u) \partial_{\bar{u}} G(\bar{u})}{(1 - \frac{\gamma}{2} FG)^2} du d\bar{u}. \end{aligned} \quad (32)$$

The corresponding scalar curvature only depends on the coordinates w, \bar{w} since in the u, \bar{u} ‘direction’ the curvature is constant due to the Liouville nature of our solution. However the space is asymptotically flat only for $\alpha = 0$. The antisymmetric tensor field strength has for example the following non-vanishing component

$$H_{u\bar{u}w} = K_{u\bar{u}w} = \gamma \frac{F'G'}{(1 - \frac{\gamma}{2} FG)^2}. \quad (33)$$

This shows that, whenever the choice of F and G is such that $G = \bar{F}$, one can use F and \bar{F} as coordinates of the four-dimensional background instead of u and \bar{u} . We will now indeed set $G = \bar{F}$. Then, one can set $F = \sqrt{2}u$ without loss of generality. We will, for later convenience, also set $\gamma = -1$ and $\alpha = -\frac{\rho^2}{2}$. Then, the resulting Kähler potential reads

$$\begin{aligned} K &= (w + \bar{w}) \ln 2 - 2(w + \bar{w}) \ln(1 + u\bar{u}) + \int^{w+\bar{w}} \ln \left(-\frac{\rho^2}{2} + \frac{1}{2}(w + \bar{w})^2 \right) \\ &= (w + \bar{w}) \left(-2 - 2\ln(1 + u\bar{u}) + \ln(-\rho^2 + (w + \bar{w})^2) \right) + 2\rho \operatorname{artanh}\left(\frac{w + \bar{w}}{\rho}\right) \end{aligned} \quad (34)$$

Finally, performing the change $w \rightarrow -w$ yields

$$K = (w + \bar{w}) \left(2 + 2\ln(1 + u\bar{u}) - \ln(-\rho^2 + (w + \bar{w})^2) \right) - 2\rho \operatorname{artanh}\left(\frac{w + \bar{w}}{\rho}\right) \quad (35)$$

We now proceed to show that the dual geometry, obtained from the Kähler potential dual to (35), describes an Eguchi-Hanson instanton.³ It is important to emphasize that the duality transformation will not destroy the $N = 2$ world-sheet supersymmetry of the original model, since it can be performed within the $N = 2$ superfield formalism. The dual Kähler potential, $\tilde{K}(u, \bar{u}, \psi + \bar{\psi})$, is obtained by performing an abelian duality transformation [18] with respect to the U(1) Killing symmetry of the Kähler potential $K(u, \bar{u}, w + \bar{w})$. Alternatively, the dual Kähler potential $\tilde{K}(U, \bar{U}, \Psi + \bar{\Psi})$ can also be obtained [6] by a Legendre transformation of $K(U, \bar{U}, W + \bar{W})$, which amounts to replacing the twisted chiral superfield W (whose lowest component field is w) by a chiral superfield Ψ (whose lowest component field is ψ), as follows [16]. Starting from

$$\tilde{K}(w + \bar{w}, u, \bar{u}, \psi + \bar{\psi}) = K(u, \bar{u}, w + \bar{w}) - (w + \bar{w})(\psi + \bar{\psi}) \quad (36)$$

³This dual geometry will also exhibit the SL(2,C) symmetry of the original geometry

and demanding that

$$\partial_w \tilde{K} = \partial_w K - (\psi + \bar{\psi}) = 0 \quad (37)$$

allows one to solve for $w + \bar{w} = g(u, \bar{u}, \psi, \bar{\psi})$ as a function of u, \bar{u}, ψ and $\bar{\psi}$. Then, reinserting $w + \bar{w} = g(u, \bar{u}, \psi, \bar{\psi})$ into (36) yields the dual Kähler potential $\tilde{K}(u, \bar{u}, \psi, \bar{\psi})$.

The dual background metric is then given by

$$\tilde{G}_{\mu\nu} = \begin{pmatrix} 0 & \tilde{K}_{u\bar{u}} & 0 & \tilde{K}_{u\bar{\psi}} \\ \tilde{K}_{u\bar{u}} & 0 & \tilde{K}_{\psi\bar{u}} & 0 \\ 0 & \tilde{K}_{\psi\bar{u}} & 0 & \tilde{K}_{\psi\bar{\psi}} \\ \tilde{K}_{u\bar{\psi}} & 0 & \tilde{K}_{\psi\bar{\psi}} & 0 \end{pmatrix}. \quad (38)$$

The dual antisymmetric tensor field background vanishes, since the dual Kähler potential \tilde{K} is now only given in terms of chiral superfields. Finally, the dual dilaton field is given as

$$2\tilde{\Phi} = 2\Phi - \ln 2K_{w\bar{w}} \quad (39)$$

For the Kähler potential (35) under consideration, it follows from (14) that the dual dilaton is constant, $\tilde{\Phi} = \text{const.}$ Thus, the dual geometry associated with the dual of (35) is entirely described by a purely gravitational background with Ricci flat background metric (38), which we will compute in the following. Inserting the Kähler potential (35) into (37) gives $w + \bar{w} = g(u, \bar{u}, \psi, \bar{\psi})$ as

$$w + \bar{w} = \sqrt{\rho^2 + e^{-(\psi+\bar{\psi})} (1 + u\bar{u})^2} \quad (40)$$

Then, inserting (40) into (36) yields the dual Kähler potential as

$$\tilde{K}(u, \bar{u}, \psi, \bar{\psi}) = 2g(u, \bar{u}, \psi, \bar{\psi}) - 2\rho \operatorname{artanh} \frac{g(u, \bar{u}, \psi, \bar{\psi})}{\rho} \quad (41)$$

Inspection of (40) shows that a natural set of holomorphic coordinates is given by

$$\begin{aligned} z_1 &= e^{\frac{-\psi}{2}} \\ z_2 &= e^{\frac{-\psi}{2}} u \end{aligned} \quad (42)$$

such that

$$g(z_1, z_2) = \sqrt{\rho^2 + (z_1 \bar{z}_1 + z_2 \bar{z}_2)^2} \quad (43)$$

After setting

$$Q = z_1 \bar{z}_1 + z_2 \bar{z}_2 \quad (44)$$

one computes that

$$\frac{d\tilde{K}}{dQ} = 2 \frac{\sqrt{\rho^2 + Q^2}}{Q} \quad (45)$$

which indeed describes the Kähler potential [19] for the usual hyperkähler Eguchi-Hanson metric. The background metric associated with (41) is, in coordinates (42), then given by [19]

$$g_{1\bar{1}} = 2 \left(\frac{g}{Q^2} |z_2|^2 + \frac{1}{g} |z_1|^2 \right), \quad g_{2\bar{2}} = 2 \left(\frac{g}{Q^2} |z_1|^2 + \frac{1}{g} |z_2|^2 \right), \quad g_{1\bar{2}} = -2 \frac{\rho^2}{g Q^2} z_2 \bar{z}_1 \quad (46)$$

Since this selfdual metric is hyperkähler, the $N = 4$ world-sheet supersymmetry is guaranteed.

In summary, we have exhibited the relation of four-dimensional $N = 2, 4$ supersymmetric, non-Kählerian superstring backgrounds with one Abelian Killing symmetry to a particularly simple integrable system, namely the Liouville equation respectively continual Toda equation for the case of vanishing central charge deficit. The solutions of the Liouville resp. Toda equations lead to a new class of metric, $B_{\mu\nu}$ and Φ backgrounds which can be explicitly constructed. Via the duality transformation, a particular subset of these solutions is equivalent to the well-known Eguchi-Hanson instanton in four dimensions. We believe that it is interesting to further investigate the relation of integrable systems to non-trivial (super) string backgrounds.

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